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Regular approximation of singular second-order differential expressions

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Abstract

In this paper we construct regular real self-adjoint approximations for real self-adjoint operators associated with the differential expression

$$\ell(y)=\frac{1}{w}\big[-(py')'+qy\big].$$

If 0 is in the resolvent of the original operator, then the construction guarantees that 0 is a point of the resolvent set of the approximating operators. The notion of strong resolvent convergence is generalized and we prove the strong resolvent convergence of the approximations.

1. Introduction

We consider formally self-adjoint differential expressions of the form

$$\ell(y)=\frac{1}{w}\big[-(py')'+qy\big] \tag{1}$$

defined on an interval $I=(a,b)$, $-\infty\leqslant a<b\leqslant\infty$, with the standard minimal assumptions [4] on the real data p, q, and w,

$$1/p,q,w\in L_{\mathrm{loc}}(I),\qquad w(t)>0\text{ a.e.} \tag{2}$$

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The expression $\ell(y)$ is said to be *regular* if (a, b) is finite and the functions $1/p, q, w$ are integrable in the whole interval (a, b). Otherwise, $\ell(y)$ is said to be *singular*. In particular, the left end-point is *regular* if $a > -\infty$ and if $1/p, q, w$ belong to $L(a, \beta)$ for every $\beta < b$; otherwise we say that the endpoint a is *singular*. Similarly, we define the regularity and singularity of the right end-point b. Obviously, the expression $\ell(y)$ is regular if and only if both end-points a and b are regular.

Our interest in this work is approximating the solution of the "possibly" singular direct problem

$$\ell(u) = f \tag{3}$$

by a neighboring regular one.

The idea of approximating singular problems by regular ones was studied in [1] in the course of approximating eigenvalues of Sturm–Liouville problems. While the construction of the regular operators was adequate for their purpose, it is not the case for the direct problem (3). This is because the construction in [1] allows only for convergence with respect to nonreal resolvent values.

What we need for the direct problem (3) is convergence with respect to the resolvent point 0. It turns out that the construction in the case of a real resolvent point is a nontrivial modification of the approach in [1]. This is the main task we accomplish in this paper. The natural assumption we make here is the invertibility of the operators associated with the formal expression ℓ, i.e., we will always assume that 0 is a point of the resolvent set. The theoretical background needed here can be found in [4–6].

The content of this paper is as follows. After this introduction we present some preliminaries. Then we introduce the so-called symmetric restrictions of the self-adjoint operator associated with ℓ. These symmetric operators are then extended to particularly constructed self-adjoint operators. Properties of convergence of these operators are then studied. Finally, we give examples to illustrate the construction process of the extensions.

2. Preliminaries

In this section we introduce notation, definitions, and known results. For more details, we refer the reader to [4,6].

Let $\mathcal{H} = L^2_w(I)$ and $y^{[1]}$ denote the quasi-derivative of y defined by

$$y^{[1]} = py'. \tag{4}$$

The formally self-adjoint differential expression $\ell(y)$ generates the following operators in $\mathcal{H}$. The *maximal operator* L is defined by

$$\begin{aligned} &\mathcal{D}(L) = \mathcal{D} = \bigl\{y \in \mathcal{H}\colon\ y, y^{[1]} \in AC_{\mathrm{loc}}(I) \text{ and } \ell(y) \in \mathcal{H}\bigr\}, \\ &Ly = \ell(y), \quad y \in \mathcal{D}. \end{aligned} \tag{5}$$

Since $\mathcal{D}$ is dense in $\mathcal{H}$, L has a uniquely defined adjoint. Let

$$L_0 = L^* \quad \text{and} \quad \mathcal{D}_0 = \text{domain of } L_0.$$

The operator L_0 is called the *minimal operator* generated by ℓ and it is known [4] that $\mathcal{D}_0 \subseteq \mathcal{D}$, $\mathcal{D}_0$ is dense in $\mathcal{H}$, and $L_0^* = L$. In other words, $L_0 \subset L = L_0^*$ and thus L_0 is a symmetric closed operator. Moreover, any self-adjoint extension S of L_0 is a self-adjoint restriction of L and vice versa, i.e.,

$$L_0 \subset S = S^* \subset L_0^* = L.$$

The minimal operator L_0 is the closure of the symmetric operator L_0' defined by

$$\mathcal{D}_0' = \{y \in \mathcal{D}\colon\ y \text{ has compact support in} \operatorname{int}(I)\},$$
$$Ly = \ell(y), \quad y \in \mathcal{D}_0'.$$

The operator L_0' is called the *pre-minimal operator* of ℓ on I.

For a fixed nonreal λ, let $\mathcal{R}_\lambda$ denote the range of $L_0 - \lambda E$, where E is the identity operator on $\mathcal{H}$. The deficiency space $\mathcal{N}_\lambda$ of L_0 is defined to be the orthogonal complement of $\mathcal{R}_\lambda$ in $\mathcal{H}$, i.e.,

$$\mathcal{H} = \mathcal{N}_\lambda \oplus \mathcal{R}_\lambda.$$

It is shown in [4] that

$$\mathcal{N}_\lambda = \{y \in \mathcal{H}\colon\ L_0^* y = Ly = \bar{\lambda} y\},$$
$$\mathcal{D} = \mathcal{D}_0 \dotplus \mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}},$$

and

$$\dim(N_\lambda) = \dim(N_{\bar{\lambda}}).$$

The symbol $\dotplus$ denotes direct sum of not necessarily orthogonal spaces. We denote the common value $\dim(N_\lambda)$ by d and call d the deficiency index of L_0 on I. It is shown in [4] that $0 \leqslant d \leqslant 2$, and if one end-point is regular and the other is singular, then $1 \leqslant d \leqslant 2$.

For $y, z \in \mathcal{D}$ and $x \in I$ define the sesquilinear form

$$[y, z](x) = y(x)\bar{z}^{[1]}(x) - \bar{z}(x) y^{[1]}(x). \tag{6}$$

Note that the limits of the terms in (6) as $x \to a^+, b^-$ exist. We denote these limits by

$$\lim_{x \to a^+} [y, z](x) = [y, z](a), \qquad \lim_{x \to b^-} [y, z](x) = [y, z](b).$$

Also, let

$$[y, z]_\alpha^\beta = [y, z](\beta) - [y, z](\alpha). \tag{7}$$

The following characterizations of $\mathcal{D}_0$ are proven in [4].

Proposition 1. *In general*

$$\mathcal{D}_0 = \{y \in \mathcal{D}\colon\ [y, z]_a^b = 0,\ \forall z \in \mathcal{D}\}. \tag{8}$$

If a and b are regular, then

$$\mathcal{D}_0 = \{y \in \mathcal{D}\colon\ y(a) = y^{[1]}(a) = y(b) = y^{[1]}(b) = 0\}. \tag{9}$$

Any self-adjoint extension S of L_0 is characterized by a unitary transformation $U:\mathcal{N}_{\bar{\lambda}} \to \mathcal{N}_{\lambda}$ such that

$$\mathcal{D}(S) = \mathcal{D}_0 \dotplus (U + I)\mathcal{N}_{\bar{\lambda}}. \tag{10}$$

The following lemma characterizes the domain of definition of the self-adjoint extension S by means of boundary conditions.

Lemma 2. *Suppose the deficiency index of the operator L_0 is d, where $1 \leqslant d \leqslant 2$. Given any self-adjoint extension S of L_0, with domain $\mathcal{D}(S)$, $\mathcal{D}_0 \subset \mathcal{D}(S) \subset \mathcal{D}$, there exists $\{\eta_1, \ldots, \eta_d\}$ in $\mathcal{D}(S)$ satisfying*

(i) $\eta_1, \ldots, \eta_d$ *are linearly independent modulo* $\mathcal{D}_0$;
(ii) $[\eta_i, \eta_j]_a^b = 0$, $1 \leqslant i, j \leqslant d$;
(iii) $\mathcal{D}(S) = \{y \in \mathcal{D}\colon [y, \eta_i]_a^b = 0,\ 1 \leqslant i \leqslant d\}$.

Conversely, the above three properties define the domain of a self-adjoint extension S of L_0.

Note that in view of the this lemma we can write

$$\mathcal{D}(S) = \mathcal{D}_0 \dotplus \operatorname{span}[\eta_1, \ldots, \eta_d],$$

i.e., $\mathcal{D}(S)$ is a d-dimensional extension of $\mathcal{D}_0$.

Real self-adjoint extensions of L_0 are characterized as follows (see [3]).

Lemma 3. *A self-adjoint extension S of L_0 is real if and only if the functions $\eta_1, \ldots, \eta_d$ of Lemma 2 can be chosen to be real and $[\eta_i, \eta_j]_\alpha^\beta = 0$ for all α sufficiently close to a and β sufficiently close to b.*

We remark here that if $d = 0$ then L_0 is a real self-adjoint operator and the above lemma is vacuous. For the rest of this paper we use S to denote a real self-adjoint extension of L_0.

For a given operator T, we denote its resolvent set by $\rho(T)$ and its spectrum by $\sigma(T)$. The point spectrum, residual spectrum, and continuous spectrum are denoted by $\sigma_p(T)$, $\sigma_r(T)$, and $\sigma_c(T)$, respectively. The set $\rho_\sigma(T) = \rho(T) \cup \sigma_r(T)$ is the set of points of regular type which are the points $\tau \in \mathbb{C}$ such that

$$\|(T - \tau)u\| \geqslant \kappa \|u\|, \tag{11}$$

for some $\kappa > 0$ and all u in the domain of T. Note that for self-adjoint operators, σ_r is empty and thus $\rho_\sigma = \rho$. We also note here that the minimal operator with deficiency index $\geqslant 1$ has no resolvent points.

3. Induced symmetric operators and their extensions

In this section we consider the operators generated by restricting ℓ to subintervals with regular endpoints. We extend the minimal domain to a domain of symmetric operators which we extend further to a domain of self-adjoint operators.

Let $I^r = (a^r, b^r)$ with $-\infty \leqslant a \leqslant a^r < b^r \leqslant b \leqslant \infty$, $r \in \mathbb{N} = \{1, 2, \ldots\}$, such that $a^r \to a$ and $b^r \to b$. If a (b) is singular then we assume that $a < a^r$ $(b^r < b)$. Let $\mathcal{H}^r = L^2_w(I^r)$. For each interval I_r, let L^r be the maximal operator defined by

$$\mathcal{D}(L^r) = \mathcal{D}^r = \{y \in \mathcal{H}^r\colon\ y, y^{[1]} \in AC_{\text{loc}}(I^r) \text{ and } \ell(y) \in \mathcal{H}^r\}, \tag{12}$$

and

$$L^r y = \ell(y), \quad y \in \mathcal{D}^r.$$

Let L_0^r and $\mathcal{D}_0$ denote the minimal operator and its domain.

We have the following remark.

Remark 4. The differential expression ℓ of (1) is regular on I^r. Therefore, the minimal operator L_0^r always has deficiency index 2 and, consequently, any self-adjoint operator S^r associated with ℓ is a 2-dimensional extension of L_0^r. On the other hand, from Lemma 2, if the minimal operator L_0 has deficiency index d, $0 \leqslant d \leqslant 2$, then any self-adjoint operator S, $L_0 \subset S = S^* \subset L$, is a d-dimensional extension of L_0.

Assume now that we are given a real self-adjoint extension S of L_0 such that

$$\text{(assumption)} \quad 0 \in \rho(S). \tag{13}$$

We will show in this section that, in general, S induces a certain closed symmetric operator $\tilde{S}^r$ in $\mathcal{H}^r$ for r sufficiently large. This symmetric operator will then be extended to a self-adjoint operator S^r in $\mathcal{H}^r$. We will show in the next section that under our construction, for r sufficiently large, $0 \in \rho_\sigma(\tilde{S}^r)$ and thus $0 \in \rho(S^r)$. The sense as well as the properties of convergence of the operators S^r to S will also be introduced and investigated in the next section.

In the next theorem we construct a closed symmetric extension of L_0^r.

Theorem 5. *Suppose the deficiency index of the operator L_0 is d, $0 \leqslant d \leqslant 2$. Let S be a real self-adjoint extension of L_0 defined by the functions $\eta_1, \ldots, \eta_d$ of Lemma 3 and denote their restrictions to I^r by $\eta_1^r, \ldots, \eta_d^r$. Then, for sufficiently large r,*

(i) *$\eta_1^r, \ldots, \eta_d^r$ are linearly independent modulo $\mathcal{D}_0^r$;*
(ii) *the operator $\tilde{S}^r$ defined by ℓ on the domain*

$$\tilde{\mathcal{D}}^r = \mathcal{D}_0^r \dotplus \operatorname{span}[\eta_1^r, \ldots, \eta_d^r]$$

is a closed symmetric extension of L_0^r.

Proof. (i) Although this follows from the way the functions $\eta_1, \ldots, \eta_d$ were defined in [3], we prefer to give an alternative proof here. Suppose the statement is false. Then we will be able to find a sequence $\{\alpha^r\} \subset \mathbb{C}^d$, $r \in \mathbb{N}_1 \subset \mathbb{N}$, such that $\xi^r = \sum_{i=1}^d \alpha_i^r \eta_i^r \in \mathcal{D}_0^r$. We may assume without loss of generality that $\|\alpha^r\|_\infty = 1$. Let ξ_0^r be defined by

$$\xi_0^r(x) = \begin{cases} \xi^r(x), & x \in I^r, \\ 0, & x \in I \setminus I^r. \end{cases} \tag{14}$$

Then, there is a vector $\alpha \in \mathbb{C}^d$,

$$\|\alpha\|_\infty = 1, \tag{15}$$

and a subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ on which $\xi_0^r \to \xi = \sum_{i=1}^d \alpha_i \eta_i$. We will show now that $\xi \in \mathcal{D}_0$. Let $f \in \mathcal{D}$ and note that the restriction of f to I^r belongs to $\mathcal{D}^r$ (see (5)). Hence $[f, \xi^r]_{a^r}^{b^r} = 0$ for all $r \in \mathbb{N}_2$. Passing to the limit in $\mathbb{N}_2$ gives $[f, \xi]_a^b = 0$. Hence, $\xi \in \mathcal{D}_0$. On the other hand, since $\eta_1, \dots, \eta_d$ are linearly independent modulo $\mathcal{D}_0$ we get $\alpha_i = 0$, $i = 1, \dots, d$. This contradicts (15).

(ii) Clearly, $\tilde{S}^r$ is an extension of L_0^r. $\tilde{S}^r$ is closed since it is a finite-dimensional extension of L_0^r. The symmetry of $\tilde{S}^r$ is a consequence of the following:

$$\begin{aligned}
\langle \tilde{S}^r \eta_i^r, \eta_j^r \rangle &= \langle \ell(\eta_i^r), \eta_j^r \rangle = [\eta_i^r, \eta_j^r]_{a^r}^{b^r} + \langle \eta_i^r, \ell(\eta_j^r) \rangle \\
&= \langle \eta_i^r, \ell(\eta_j^r) \rangle \quad \text{(see Lemma 3)} \\
&= \langle \eta_i^r, \tilde{S}^r \eta_j^r \rangle, \quad i, j = 1, \dots, d.
\end{aligned}$$

Also for any $u \in \mathcal{D}_0^r$,

$$\begin{aligned}
\langle \tilde{S}^r u, \eta_j^r \rangle &= \langle \ell(u), \eta_j^r \rangle = [u, \eta_j^r]_{a^r}^{b^r} + \langle u, \ell(\eta_j^r) \rangle \\
&= \langle u, \ell(\eta_j^r) \rangle \quad \text{(by the definition of } \mathcal{D}_0^r) \\
&= \langle u, \tilde{S}^r \eta_j^r \rangle, \quad j = 1, \dots, d. \qquad \square
\end{aligned}$$

It follows from Theorem 5 that $\tilde{S}^r$ is a d-dimensional extension of L_0^r and thus from Remark 4 we have

$$\mathcal{D}_0^r \subset \tilde{\mathcal{D}}^r \subset \mathcal{D}^r. \tag{16}$$

Consequently, if $d = 2$, the operators $S^r = \tilde{S}^r$ are self-adjoint. Otherwise, for $d < 2$, we need to construct an appropriate subspace of dimension $2 - d$ in $\mathcal{D}^r \setminus \tilde{\mathcal{D}}^r$.

Now we proceed to construct self-adjoint extensions S^r of $\tilde{S}^r$ for $d < 2$. To achieve that, we employ the general theory of symmetric extensions of symmetric operators (see [4]). Let $\mathcal{N}_i^r, \mathcal{N}_{-i}^r$ be the deficiency spaces associated with L_0^r. Then $\dim(\mathcal{N}_i^r) = \dim(\mathcal{N}_{-i}^r) = 2$. Let

$$\mathcal{N}_i^r = \operatorname{span}[\varphi_1^r, \varphi_2^r], \qquad \mathcal{N}_{-i}^r = \operatorname{span}[\bar{\varphi}_1^r, \bar{\varphi}_2^r]. \tag{17}$$

Define the unitary transformation $U : \mathcal{N}_{-i}^r \to \mathcal{N}_i^r$ by

$$U \bar{\varphi}_i^r = \varphi_i^r, \quad i = 1, 2,$$

and define the space

$$\mathcal{M}^r = (U + I)\mathcal{N}_{-i}^r = \operatorname{span}[\xi_1^r, \xi_2^r],$$

where

$$\xi_i^r = \varphi_i^r + \bar{\varphi}_i^r, \quad i = 1, 2. \tag{18}$$

Note that $\mathcal{M}^r$ is a 2-dimensional space in $\mathcal{D}^r \setminus \mathcal{D}_0^r$. Thus when $d = 0$ we can construct a self-adjoint extension S^r of L_0^r by

$$\hat{\mathcal{D}}^r = \mathcal{D}_0^r \dotplus \mathcal{M}^r, \qquad S^r u = L^r u, \quad u \in \hat{\mathcal{D}}^r.$$

The next lemma establishes the property that $\mathcal{M}^r \cap \ker(L^r) = \{0\}$. This implies that 0 is not an eigenvalue of S^r. This will be used in the next section to show that actually $0 \in \rho(S^r)$.

Lemma 6. *The equation*

$$\ell(y) = 0$$

does not have any nontrivial solutions in $\mathcal{M}^r$.

Proof. Suppose

$$\ell\big(\alpha_1 \xi_1^r + \alpha_2 \xi_2^r\big) = 0.$$

Then

$$\alpha_1\big(i\varphi_1^r - i\bar{\varphi}_1^r\big) + \alpha_2\big(i\varphi_2^r - i\bar{\varphi}_2^r\big) = 0.$$

Rearranging, we get

$$\big(\alpha_1\varphi_1^r + \alpha_2\varphi_2^r\big) - \big(\alpha_1\bar{\varphi}_1^r + \alpha_2\bar{\varphi}_2^r\big) = 0.$$

Since $\mathcal{N}_i^r, \mathcal{N}_{-i}^r$ are linearly independent, $(\alpha_1\varphi_1^r + \alpha_2\varphi_2^r) = (\alpha_1\bar{\varphi}_1^r + \alpha_2\bar{\varphi}_2^r) = 0$. And since φ_1^r, φ_2^r are linearly independent, then $\alpha_1 = \alpha_2 = 0$. $\square$

For the self-adjoint extension S^r of $\tilde{S}^r$ in the case $d = 1$ we need to construct an appropriate 1-dimensional space in $\mathcal{D}^r \setminus \tilde{\mathcal{D}}^r$. For that we need the following lemma.

Lemma 7. *Suppose* $d = 1$. *Let* $\mathcal{N}^r = \operatorname{span}[\eta^r]$, *where* η^r *is as defined in Theorem* 5. *Then there is a function* $\xi^r \in \mathcal{M}^r$ *which is orthogonal to* $\mathcal{N}^r$.

Proof. Since by construction, $\mathcal{N}^r \subset \mathcal{M}^r$, we can write ξ_1, ξ_2 in (18) as

$$\xi_1^r = h_1 + \alpha\eta^r, \qquad \xi_2^r = h_2 + \beta\eta^r,$$

where $h_1, h_2 \in \mathcal{M}^r \ominus \mathcal{N}^r$ and $\alpha, \beta \in \mathbb{C}$. If $\alpha = 0$ $(\beta = 0)$ then $\xi^r = \xi_1^r$ $(\xi^r = \xi_2^r)$ is orthogonal to $\mathcal{N}^r$. Otherwise the function $\xi^r = \xi_1^r/\alpha - \xi_2^r/\beta$ is orthogonal to $\mathcal{N}^r$. $\square$

It follows from this lemma that, if $d = 1$, then the operator S^r defined in $\mathcal{H}^r$ by

$$\hat{\mathcal{D}}^r = \tilde{\mathcal{D}}^r \dotplus \operatorname{span}[\xi^r], \qquad S^r u = L^r u, \quad u \in \hat{\mathcal{D}}^r,$$

is a self-adjoint extension of L_0^r. To see this, write $\xi^r = \alpha\xi_1^r + \beta\xi_2^r = (U + I)(\alpha\bar{\varphi}_1^r + \beta\bar{\varphi}_2^r)$. Then $\operatorname{span}[\xi^r]$ corresponds to the restriction of $(U + I)$ to the subspace of $\mathcal{N}_{-i}^r$ spanned by the function $\alpha\bar{\varphi}_1^r + \beta\bar{\varphi}_2^r$. Since ξ^r does not belong to $\tilde{\mathcal{D}}^r$ from the theory of extensions of symmetric operators [4] we get that $\hat{\mathcal{D}}^r$ is the domain of a symmetric extension of $\tilde{S}^r$. Furthermore, since the deficiency index of $\tilde{S}^r$ is 1 (being a 1-dimensional extension of L_0^r), it follows that the deficiency index of S^r is zero. Hence, S^r is self-adjoint.

4. Convergence of the induced operators

In this section we discuss the convergence properties of the real self-adjoint extensions S^r of L_0^r, constructed in the previous section, to the real self-adjoint extensions S of L_0. For this purpose we define the following operators. Define the extension operator $E^r : \mathcal{H}_r \to \mathcal{H}$ by

$$E^r f(x) = \begin{cases} f(x), & x \in I^r, \\ 0, & x \in I \setminus I^r. \end{cases} \tag{19}$$

Define the restriction operator $R^r : \mathcal{H} \to \mathcal{H}^r$ by

$$R^r f = f|_{I^r}. \tag{20}$$

We will establish in this section that

(1) $\{S^r\}$ converges strongly to S, i.e., there is a core $\mathcal{C}(S)$ of the operator S such that given $u \in \mathcal{C}(S)$ we can find a sequence $u^r \in \hat{\mathcal{D}}^r$ such that $E^r u^r \to u$ and $E^r S^r u^r \to Su$. This will be denoted by $S^r \xrightarrow{s} S$.
(2) $\{S^r\}$ converges to S in the strong resolvent sense, i.e., $E^r (S^r)^{-1} R^r f \to S^{-1} f$ for all $f \in \mathcal{H}$. This will be denoted by $S^r \xrightarrow{sr} S$.

It will then follow that

(3) $\{S^r\}$ converges to S in the strongly stable sense, i.e., $S^r \xrightarrow{s} S$ and $\{(S^r)^{-1}\}$ is uniformly bounded. This will be denoted by $S^r \xrightarrow{ss} S$.

We begin with the following lemma which gives some elementary properties of the operators E^r and R^r.

Lemma 8. *Let E^r and R^r be the operators defined by* (19) *and* (20)*, respectively. Then*

(1) *E^r is an isometry and R^r has norm* 1;
(2) *$(E^r)^* = R^r$ and $(R^r)^* = E^r$*;
(3) *$R^r E^r : \mathcal{H}^r \to \mathcal{H}^r$ is the identity*;
(4) *$P^r = E^r R^r$ is an orthogonal projection in $\mathcal{H}$ and $P^r \xrightarrow{s} I$*;
(5) *$E^r L_0^r = L_0' E^r$ on $\mathcal{D}_0^r$ and, by taking conjugates*;
(6) *$L^r R^r = R^r L$ on $\mathcal{D}(S)$.*

The next theorem establishes the strong convergence $S^r \xrightarrow{s} S$.

Theorem 9. $S^r \xrightarrow{s} S$.

Proof. Let

$$\mathcal{C}(S) = \mathcal{D}_0' \dotplus \operatorname{span}[\eta_1, \ldots, \eta_d],$$

where η_i are as in Lemma 3. Let $u \in \mathcal{C}(S)$ and write $u = u_0 + u_1$, $u_0 \in \mathcal{D}_0'$ and $u_1 \in \operatorname{span}[\eta_1, \dots, \eta_d]$. It follows from Lemma 3 that there is $N_1 \in \mathbb{N}$ such that $R^r u_1 \in \operatorname{span}[\eta_1^r, \dots, \eta_d^r]$ for all $r \geqslant N_1$. Also there is $N_2 \in \mathbb{N}$ such that $R^r u_0 \in \mathcal{D}_0^r$ for all $r \geqslant N_2$. It follows that $R^r u \in \tilde{\mathcal{D}}^r \subset \hat{\mathcal{D}}^r$ for all $r \geqslant N = \max\{N_1, N_2\}$. Define $u^r \in \tilde{\mathcal{D}}^r$ by

$$u^r = \begin{cases} 0, & r < N, \\ R^r u, & r \geqslant N, \end{cases}$$

then, for $r \geqslant N$, $E^r u^r = P^r u \to u$ and

$$\begin{aligned} E^r S^r u^r &= E^r \tilde{S}^r u^r = E^r L_0^r R^r u_0 + E^r L^r R^r u_1 \\ &= L_0' E^r R^r u_0 + E^r R^r L u_1 = L_0' u_0 + P^r S u_1 \to L_0' u_0 + S u_1 = S u. \qquad \square \end{aligned}$$

To establish the strong resolvent convergence, we must first show that $0 \in \rho(S^r)$. To do this we begin by showing that, for sufficiently large r, $0 \in \rho_\sigma(\tilde{S}^r)$. The following three lemmas are needed for that purpose.

Lemma 10. *Let $\mathcal{M} = \operatorname{span}[\eta_1, \dots, \eta_d]$, η_i as in Lemma 3 ($\mathcal{M} = \{0\}$ if $d = 0$). Then $\mathcal{R}(L_0) \cap S\mathcal{M} = \{0\}$.*

Proof. Suppose that $L_0 u_0 = S u_1$ for some $u_0 \in \mathcal{D}_0$ and $u_1 \in \mathcal{M}$. Since $L_0 \subset S$, $S(u_0 - u_1) = 0$. Since S is one to one, then $u_0 - u_1 = 0$. Since $\mathcal{D}_0$ and $\mathcal{M}$ are linearly independent, $u_0 = u_1 = 0$. $\square$

Lemma 11. *Let X, Y be closed subspaces such that $X \cap Y = \{0\}$ and Y is finite dimensional. The two sets of vectors $x \in X$ and $y \in Y$ such that $\|x + y\| = 1$ are bounded.*

Proof. If the statement were false, then at least one of the two sets is unbounded. But the equality implies that the other set must also be unbounded. Therefore, there exist two sequences $x_n \in X$ and $y_n \in Y$ such that $\|x_n + y_n\| = 1$ and $x_n \to \infty$, $y_n \to \infty$. Let $u_n = x_n/\|y_n\|$ and $v_n = y_n/\|y_n\|$. Since Y is finite dimensional, we may assume that $v_n \to v$, and this implies that $\|v\| = 1$. Now $\|u_n + v_n\| = 1/\|y_n\| \to 0$. Hence, $u_n + v_n \to 0$. Therefore, $u_n \to -v$. Since X, Y are closed, we get $v \in X \cap Y$ and therefore $v = 0$. Thus $1 = \|v\| = 0$, a contradiction. $\square$

Lemma 12. *There exists $\gamma > 0$ such that $\|L_0 u + P^r S v\| \geqslant \gamma \|L_0 u + S v\|$ for all $u \in \mathcal{D}_0$, $v \in \mathcal{M}$, and all r sufficiently large.*

Proof. If not, then there exist two sequences $u^r \in \mathcal{D}_0$, $v^r \in \mathcal{M}$ such that $\|L_0 u^r + S v^r\| = 1$ but $\|L_0 u^r + P^r S v^r\| \to 0$. Since $\mathcal{R}(L_0)$ and $S\mathcal{M}$ are closed subspace and $\mathcal{M}$ is finite dimensional, it follows from Lemmas 10 and 11 that $\|S v^r\|$ is bounded. And since $0 \in \rho(S)$, $\|v^r\|$ is bounded. Since $\mathcal{M}$ is finite dimensional, we may assume that $v^r \to v$ and $S v^r \to S v$. Also $P^r S v^r \to S v$. This can be seen as follows:

$$\begin{aligned} \|P^r S v^r - S v\| &\leqslant \|P^r S v^r - P^r S v\| + \|P^r S v - S v\| \\ &\leqslant \|S v^r - S v\| + \|P^r S v - S v\| \to 0. \end{aligned}$$

Together with $L_0u^r + P^r Sv^r \to 0$ we get $L_0u^r \to Sv$. Since $\mathcal{R}(L_0)$ is closed, $Sv \in \mathcal{R}(L_0)$. From Lemma 10 we get $Sv = 0$ and hence, $L_0u^r \to 0$. Thus $1 = \|L_0u^r + Sv^r\| \to 0$, a contradiction. □

We are now ready to prove that 0 is a point of regular type for $\tilde{S}^r$ for sufficiently large r.

Theorem 13. *For r sufficiently large,* $0 \in \rho_\sigma(\tilde{S}^r)$.

Proof. For $u \in \tilde{\mathcal{D}}^r$, write $u = u_0 + u_1$, $u_0 \in \mathcal{D}_0^r$, and $u_1 = \sum_{i=1}^d \alpha_i \eta_i^r$. Let $\tilde{u}_0 = E^r u_0$, $\tilde{u}_1 = \sum_{i=1}^d \alpha_i \eta_i$, and $\tilde{u} = \tilde{u}_0 + \tilde{u}_1$. Then $u = R^r \tilde{u}$ and

$$\begin{aligned} E^r \tilde{S}^r u &= L_0' E^r u_0 + E^r \tilde{S}^r u_1 = L_0' E^r u_0 + E^r L^r R^r \tilde{u}_1 \\ &= L_0' \tilde{u}_0 + E^r R^r L \tilde{u}_1 = L_0' \tilde{u}_0 + P^r S \tilde{u}_1. \end{aligned}$$

Using Lemma 12 and the assumption $0 \in \rho(S)$ we get, for sufficiently large r,

$$\begin{aligned} \|\tilde{S}^r u\| &= \|E^r \tilde{S}^r u\| = \left\|L_0' \tilde{u}_0 + P^r S \tilde{u}_1\right\| \\ &\geqslant \gamma \|L_0 \tilde{u}_0 + S\tilde{u}_1\| = \gamma \|S\tilde{u}\| \geqslant \frac{\gamma}{\|S^{-1}\|} \|\tilde{u}\| \geqslant c\|R^r \tilde{u}\| = c\|u\|, \end{aligned}$$

where $c = \gamma / \|S^{-1}\|$. The result follows from (11). □

As a direct consequence of this theorem and the construction of the self-adjoint extensions of $\tilde{S}^r$ we get that $0 \in \rho(S^r)$. We show this in the next theorem.

Theorem 14. *For r sufficiently large,* $0 \in \rho(S^r)$.

Proof. It is well known that $\sigma(S^r)$ can differ from $\sigma(\tilde{S}^r)$ by utmost a set of $2 - d$ eigenvalues. Since $0 \notin \sigma_p(\tilde{S}^r)$, by Theorem 13 and by our construction, 0 is not an eigenvalue of S^r (see Lemma 6). Thus $0 \in \rho(S^r)$. □

We have now established the existence of the operators $(S^r)^{-1}$ which are needed for the definition of the strong resolvent convergence.

Theorem 15. $S^r \xrightarrow{sr} S$.

The proof is exactly the same as that of Theorem 6.1 of [1] and we, therefore, omit it.

This theorem and the uniform boundedness principle imply that $\|(S^r)^{-1}\|$ are uniformly bounded. Taken together with Theorem 9, we obtain

Theorem 16. $S^r \xrightarrow{ss} S$.

To summarize, given a real self-adjoint operator S associated with the differential expression ℓ, such that $0 \in \rho(S)$, we constructed a sequence of real self-adjoint operators S^r such that $0 \in \rho(S^r)$ and $\{S^r\}$ converges to S in a generalized sense of the strong resolvent convergence. The construction is carried out in two steps. First, the operator S, in general, induces a sequence of symmetric operators $\tilde{S}^r$ in $\mathcal{H}^r$ such that 0 is in the domain of regularity. Second, if necessary ($d < 2$), self-adjoint extensions S^r of the operators $\tilde{S}^r$ are constructed so that $0 \in \rho(S^r)$ and convergence is guaranteed.

5. Examples

In this section we give 3 examples (one for each of the cases $d = 0, 1, 2$) to illustrate the construction of the approximating operators.

Example 1. In this example we consider the operator

$$\ell(u) = -u'' + u$$

on the interval $I = (-\infty, \infty)$. The only solutions of the equation $\ell(u) = 0$ are e^x and e^{-x}, none of which is in H. For any $u \in \mathcal{D}_0$,

$$\langle L_0 u, u\rangle = \int (u')^2 + u^2 \geqslant \int u^2 = \|u\|^2,$$

since $uu'|_{-\infty}^{\infty} = 0$, as proven in [2]. It follows from the Cauchy–Schwarz inequality that

$$\|L_0 u\| \geqslant \|u\|,$$

and thus, according to (11), we have $0 \in \rho_\sigma(L_0) = \rho(L_0) = \rho(S)$. So assumption (13) is satisfied. Therefore, the deficiency index of L_0 is 0 and $S = L_0 = L$ is the only self-adjoint operator that can be defined for the expression ℓ.

To construct the approximating operators S^r, we take two sequences $\{a_r\}$ and $\{b_r\}$ with $a_r \to -\infty$ and $b_r \to \infty$. Then, for $x \in J_r,\ J_r = (a_r, b_r)$, we have

$$\mathcal{N}_i^r = \operatorname{span}[e^{\gamma_1 x}, e^{\gamma_2 x}], \qquad \mathcal{N}_{-i}^r = \operatorname{span}[e^{\bar{\gamma}_1 x}, e^{\bar{\gamma}_2 x}],$$

where γ_1, γ_2 are the two roots of $(1+i)$. Following the construction in Section 3, we obtain

$$\xi_i^r = e^{\gamma_i x} + e^{\bar{\gamma}_i x} = 2e^{t_i x}\cos(s_i x), \quad x \in J_r,\ i = 1, 2,$$

where $t_i = \Re(\gamma_i)$ and $s_i = \Im(\gamma_i)$. Thus the operator S^r is defined by

$$\begin{aligned} &\hat{\mathcal{D}}^r = \mathcal{D}_0^r \dotplus \operatorname{span}\bigl[\xi_1^r, \xi_2^r\bigr], \\ &S^r u(x) = L_0^r u_0(x) + \alpha e^{t_1 x}\cos(s_1 x) + \beta e^{t_2 x}\cos(s_2 x), \quad x \in J_r, \end{aligned} \tag{21}$$

where $u = u_0 + \alpha\xi_1^r + \beta\xi_2^r$.

According to Theorem 14, $0 \in \rho(S^r)$. Let us also verify this directly. Consider the equation

$$S^r u = 0.$$

Using (9) and (21), we get the two initial value problems

$$\ell(u_0) = -\bigl(\alpha e^{t_1 x}\cos(s_1 x) + \beta e^{t_2 x}\cos(s_2 x)\bigr), \qquad u_0(a_r) = u_0'(a_r) = 0,$$

and

$$u_0(b_r) = u_0'(b_r) = 0,$$

the only solution of which is $\alpha = \beta = 0$, $u_0 \equiv 0$. This implies that $u \equiv 0$. Therefore, 0 is not an eigenvalue of S^r. Since S^r has a discrete spectrum (S^r is regular), $0 \in \rho(S^r)$.

Example 2. In this example we take the expression ℓ of the previous example but defined on the interval $[0, \infty)$. In a similar fashion we can show that $0 \in \rho_\sigma(L_0)$. Since in this case, $e^{-x} \in H$, the deficiency index of L_0 is 1. According to (10), self-adjoint extensions S of L_0 are characterized by complex numbers α such that $|\alpha| = 1$, and

$$\mathcal{D}_\alpha(S) = \mathcal{D}_0 \dotplus \operatorname{span}[e^{\gamma_1 x} + \alpha e^{\bar{\gamma}_1 x}], \qquad S_\alpha u = L_0 u_0 + \delta(i e^{\gamma_1 x} - i\alpha e^{\bar{\gamma}_1 x}),$$

where γ_1 is the root of $(1 - i)$ with $\Re(\gamma_1) < 0$ and $u = u_0 + \delta(e^{\gamma_1 x} + \alpha e^{\bar{\gamma}_1 x})$. In particular, the operator S corresponding to $\alpha = 1$ is self-adjoint.

Moreover, $0 \in \rho(S)$ since, if we assume that there is $u \in \mathcal{D}(S)$ such that

$$Su = 0,$$

$Lu = Su = 0$, and thus, $u \in \ker(L)$. This implies that $u = \theta e^{-x}$, for some complex number θ. On the other hand, we can write

$$u = u_0 + 2\delta e^{tx} \cos(sx) \tag{22}$$

with $t = \Re(\gamma_1)$ and $s = \Im(\gamma_1)$. By rearranging the terms in (22), we have

$$-\delta e^{tx} \cos(sx) + \theta e^{-x} = u_0 \in \mathcal{D}_0.$$

Thus, we must have

$$\left[-\delta e^{tx} \cos(sx) + \theta e^{-x}, e^{-x}\right]_0^\infty = 0.$$

This yields $\delta = 0$. Since $0 \in \rho_\sigma(L_0)$, we also get that $u_0 = 0$. As a result, u must be 0. Therefore, 0 cannot be an eigenvalue of S.

Turning to the approximation operators, it is easy to see that the function ξ_1^r of the previous example plays now the role of the function η_1^r of Lemma 5. To extend to self-adjoint operators in H^r, we use the function ξ_2^r.

Example 3. In this example we consider the expression

$$\ell(u) = -(xu')' + \frac{\alpha^2}{x} u, \quad x \in J = (0, 1).$$

The equation

$$\ell(u) = 0$$

has the solutions $\{x^\alpha, x^{-\alpha}\}$. For $-1/2 < \alpha < 1/2$, both solutions are in H. Moreover, for $u \in \mathcal{D}_0$,

$$\langle L_0 u, u\rangle = \int x u'^2 + \frac{\alpha^2}{x} u^2 \geqslant \alpha^2 \int u^2 = \alpha^2 \|u\|^2,$$

i.e., $\|L_0 u\| \geqslant |\alpha| \|u\|$. Hence, $0 \in \rho_\sigma(L_0)$ and the deficiency index of L_0 is 2.

Following the construction in [3], we use the functions $\{x^\alpha, x^{-\alpha}\}$ to construct four functions $\{\psi_i\}_{i=1}^4 \in \mathcal{D}$ such that

(1) ψ_1 and ψ_2 vanish identically in a neighborhood of 1;
(2) $\psi_1 \equiv x^\alpha$ and $\psi_2 \equiv x^{-\alpha}$ in a neighborhood of 0;
(3) ψ_3 and ψ_4 vanish identically in a neighborhood of 0;
(4) $\psi_3 \equiv x^\alpha$ and $\psi_4 \equiv x^{-\alpha}$ in a neighborhood of 1.

Let us give, without proofs, some details about the construction of ψ_1. Let

$$f = c_1 x^\alpha + c_2 x^{-\alpha},$$

where c_1 and c_2 are to be determined later. Let u_1 be defined on $[1/2, 3/4]$ such that u_1 is a solution of

$$\ell(u) = f \quad \text{on } \left[\frac{1}{2}, \frac{3}{4}\right], \qquad u_1\left(\frac{3}{4}\right) = u_1'\left(\frac{3}{4}\right) = 0.$$

Now choose c_1 and c_2 such that $u_1'(x)$ and $xu_1'(x)$ match the corresponding values of the function x^α at $x = 1/2$. Define

$$\psi_1(x) = \begin{cases} x^\alpha, & 0 < x \leqslant \frac{1}{2}, \\ u_1(x), & \frac{1}{2} \leqslant x \leqslant \frac{3}{4}, \\ 0, & \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

It can be verified that the domain

$$\mathcal{D}_0 \dotplus \operatorname{span}[\psi_1, \psi_3 - \psi_4]$$

is the domain of a self-adjoint extension S of L_0 (which corresponds to the boundary value problem $\ell(u) = g$, $u(1) = 0$), and that $0 \in \rho(S)$. In this case the operator S induces self-adjoint operators in H^r with domains

$$\hat{\mathcal{D}}^r = \mathcal{D}_0^r \dotplus \operatorname{span}\left[\xi_1^r, \xi_2^r\right],$$

where ξ_1^r and ξ_2^r are the restrictions of ψ_1 and $\psi_3 - \psi_4$, respectively, to J_r.

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